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# U-gravity: $SL(N)$

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**ABSTRACT:** We construct a duality manifest gravitational theory for the special linear group,  $SL(N)$  with  $N \neq 4$ . The spacetime is formally extended, to have the dimension  $\frac{1}{2}N(N-1)$ , yet is *gauged*. Consequently the theory is subject to a section condition. We introduce a semi-covariant derivative and a semi-covariant ‘Riemann’ curvature, both of which can be completely covariantized after symmetrizing or contracting the  $SL(N)$  vector indices properly. Fully covariant scalar and ‘Ricci’ curvatures then constitute the action and the ‘Einstein’ equation of motion. For  $N \geq 5$ , the section condition admits duality inequivalent two solutions, one  $(N-1)$ -dimensional and the other three-dimensional. In each case, the theory can describe not only Riemannian but also non-Riemannian backgrounds.

**KEYWORDS:** String Duality, Differential and Algebraic Geometry, Classical Theories of Gravity

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## 1 Introduction

While Lorentz symmetry unifies space and time into spacetime, duality requires further extension of the spacetime [1–3]. T-duality in string theory becomes a manifest  $O(D, D)$  rotation in doubled spacetime [2–10], and so do various  $\mathcal{M}$ -theory U-dualities in extended spacetimes, including the maximal  $E_{11}$  [11–17],  $E_{10}$  [18–21] and smaller cousins [22–40] (see also [41–43] for further references).

Unlike the Lorentz symmetry unification of space and time, the duality manifest extension of the spacetime calls for the existence of seemingly unphysical ‘dual’ spacetime. One simple prescription to eliminate this unphysical feature is to let all the fields be independent of the dual coordinates, e.g. [6, 7] and ‘Generalized Geometry’ [8–10, 44–48]. More covariant method is to enforce so called a *section condition* on all the functions defined on the extended spacetime. The section condition is a differential constraint and can be solved by a certain hyper-subspace, called ‘*section*’, on which the theory is restricted to live. Duality then rotates the section in the extended spacetime. Especially, acting on an isometry direction, it may produce a new solution while the section can remain unrotated. This is the very geometric insight that has motivated [6, 7] or Double Field Theory (DFT) [49–52]. Fixing the section explicitly and parametrizing the DFT variables by Riemannian ones,

DFT may locally reduce to Generalized Geometry. Then, like T-fold [53–55], by combining diffeomorphism and  $\mathbf{O}(D, D)$  rotation as for a transition function, DFT may acquire nontrivial global aspects of non-geometry [56–60].

Further, once formulated in terms of genuine  $\mathbf{O}(D, D)$  covariant variables, DFT does not merely repackage Generalized Geometry or known supergravities, but can also describe non-Riemannian backgrounds where the notion of Riemannian metric ceases to exist even *locally* [61]. In a somewhat abstract level, the DFT-metric can be defined simply as a ‘symmetric  $\mathbf{O}(D, D)$  element’, with which (bosonic) DFT and a doubled string world-sheet action [61] still make sense. For most (“non-degenerate”) cases the DFT-metric can be parametrized by Riemannian metric,  $g_{\mu\nu}$  and Kalb-Ramond  $B$ -field, which allows DFT to describe an ordinary Riemannian gravity. But, for the other (“degenerate”) cases the DFT-metric may not admit any Riemannian interpretation, even locally! An extreme example is the DFT vacuum solution where the DFT-metric coincides with the  $\mathbf{O}(D, D)$  invariant constant metric. The doubled string action then reduces to a chiral sigma model [61], similar to [62].

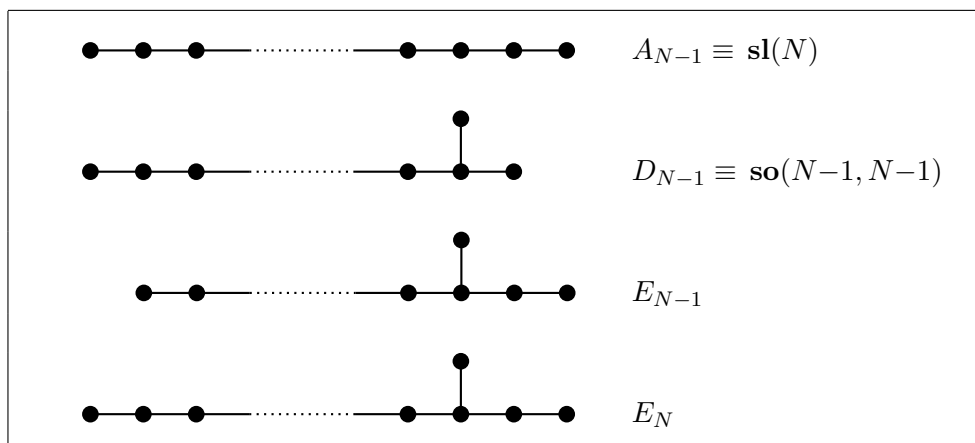
As demonstrated in refs. [57, 61], imposing the section condition is, in fact, equivalent to postulating an equivalence relation on the doubled coordinate space. That is to say, *spacetime is doubled yet gauged*. Accordingly, each equivalence class or gauge orbit represents a single physical point, and diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the gauge orbits. This allows more than one finite transformation rule of diffeomorphism [56–58]. The idea has been pushed further to construct a completely covariant string world-sheet action on doubled-yet-gauged spacetime [61], where the coordinate gauge symmetry is realized literally as one of the local symmetries of the action. In a way, understanding the section condition by gauged spacetime agrees with the lesson learned from the 20th century that ‘local symmetry dictates fundamental physics’.

In this note, we construct a duality manifest gravitational theory for the special linear group,  $\mathbf{SL}(N)$  with  $N \neq 4$ , in the name<sup>1</sup> of ‘ $\mathbf{SL}(N)$  U-gravity’. The existence of such an  $A_{N-1} \equiv \mathfrak{sl}(N)$  manifest geometry has been predicted in [63, 64] based on Dynkin diagram analyses. Thus, our construction provides an explicit realization of the prediction. Further, as seen from table 1,  $E_N$  algebra contains three inequivalent “maximal” sub-algebras,  $A_{N-1}$ ,  $D_{N-1}$  and  $E_{N-1}$ . This implies that there are three distinct ways of reducing the grand scheme of  $E_{11}$  [11–13, 15–17]: (i)  $\mathbf{SL}(11)$  U-gravity, (ii)  $\mathbf{O}(10, 10)$  DFT and (iii)  $E_{10}$  program [18–21].<sup>2</sup>

While the motivation of our work comes from the Dynkin diagram prediction [63, 64] and the  $E_{11}$  proposal [11–13, 15–17], for the actual construction of the  $\mathbf{SL}(N)$  U-gravity, we heavily employ the differential geometry tools from [57, 61, 65, 66] which were developed

<sup>1</sup>U-duality manifest theory has been occasionally dubbed, ‘Exceptional Geometry’ or ‘Exceptional Field Theory’ (EFT), e.g. [32, 33, 35–40]. However, strictly speaking, this naming should be proper only for the exceptional groups. Since our duality group,  $\mathbf{SL}(N)$ , is not exceptional, we call our theory differently. U-gravity manifests U-duality and provide a Universal framework for  $(N-1)$ -dimensional and three-dimensional gravities, as well as Riemannian and non-Riemannian geometries.

<sup>2</sup>From [63, 64], any sub-algebra of  $E_{11}$  should allow its own generalized geometry. In this context, it has been shown recently that the ‘full’ eleven-dimensional supergravity can be reformulated to manifest  $E_6$ ,  $E_7$  or  $E_8$  ‘sub’-algebras [35–38].



**Table 1.** Dynkin diagrams for  $A_{N-1}$ ,  $D_{N-1}$ ,  $E_{N-1}$  and  $E_N$ .

- *Extended-yet-gauged spacetime (section condition).*
- *Diffeomorphism generated by a generalized Lie derivative, cf. [25].*
- *Semi-covariant derivative and semi-covariant Riemann curvature.*
- *Complete covariantizations of them dictated by a projection operator.*

**Table 2.** The common features of  $\mathbf{SL}(N)$  U-gravity and DFT-geometry [57, 61, 65, 66].

to provide an underlying ‘stringy’ differential geometry for DFT (see table 2 for its characteristic).<sup>3</sup> The methods have been successfully applied to construct Yang-Mills DFT [67], coupling to fermions [68], coupling to RR-sector [69],  $\mathcal{N} = 1, D = 10$  (full order) super DFT [70],  $\mathcal{N} = 2, D = 10$  (full order) super DFT [71], and the completely covariant string action on doubled-yet-gauged spacetime [61].

Especially when  $N = 5$ , the constructed theory of U-gravity reduces to our preceding research of ‘ $\mathbf{SL}(5)$  U-geometry’ [30] cf. [22]). The present paper generalizes our previous work to an arbitrary special linear group,  $\mathbf{SL}(N)$  with  $N \neq 4$ , and also contains some novel findings, such as the semi-covariant Riemann curvature and an eight-index projection operator.

In the next section we spell out all the essential elements that constitute the  $\mathbf{SL}(N)$  U-gravity. Exposition will follow in section 3. We conclude with outlook in the final section.

## 2 Constitution of $\mathbf{SL}(N)$ U-gravity

Essential elements that constitute  $\mathbf{SL}(N)$  U-gravity are as follows.

**Notation.** Small Latin alphabet letters denote the  $\mathbf{SL}(N)$  vector indices, i.e.  $a, b, c, \dots = 1, 2, \dots, N$ .

<sup>3</sup>Alternative approaches include [72–78].

**Extended-yet-gauged spacetime.** The spacetime is formally extended, being  $\frac{1}{2}N(N-1)$ -dimensional. The coordinates carry a pair of anti-symmetric  $\mathbf{SL}(N)$  vector indices,

$$x^{ab} = -x^{ba} = x^{[ab]}, \quad (2.1)$$

and hence so does the derivative,

$$\partial_{ab} = -\partial_{ba} = \partial_{[ab]} = \frac{\partial}{\partial x^{ab}}, \quad \partial_{ab}x^{cd} = \delta_a^c \delta_b^d - \delta_a^d \delta_b^c. \quad (2.2)$$

However, *the spacetime is gauged*: the coordinate space is equipped with an equivalence relation,

$$x^{ab} \sim x^{ab} + \frac{1}{(N-4)!} \epsilon^{abc_1 \dots c_{N-4}de} \phi_{c_1 \dots c_{N-4}} \partial_{de} \varphi, \quad (2.3)$$

which we call ‘coordinate gauge symmetry’ (cf. [57, 61] for DFT analogy). In (2.3),  $\phi_{c_1 \dots c_{N-4}}$  and  $\varphi$  are arbitrary — but not necessarily covariant — functions in the theory of U-gravity. As usual,  $\epsilon^{c_1 c_2 \dots c_N}$  denotes the totally anti-symmetric Levi-Civita symbol with  $\epsilon^{12 \dots N} \equiv 1$ . Apparently, the above equivalence relation makes sense for  $N \geq 5$ . For  $N = 2, 3$ , the spacetime is not to be gauged.

Each equivalence class, or gauge orbit defined by the equivalence relation (2.3), represents a single physical point, and diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the gauge orbits.

**Realization of the coordinate gauge symmetry.** The equivalence relation (2.3) is realized in U-gravity by enforcing that, arbitrary functions and their arbitrary derivatives, denoted here collectively by  $\Phi$ , are invariant under the coordinate gauge symmetry *shift*,

$$\Phi(x + \Delta) = \Phi(x), \quad \Delta^{ab} = \frac{1}{(N-4)!} \epsilon^{abc_1 \dots c_{N-4}de} \phi_{c_1 \dots c_{N-4}} \partial_{de} \varphi. \quad (2.4)$$

**Section condition.** The invariance under the coordinate gauge symmetry (2.4) is, in fact, equivalent to a section condition,

$$\partial_{[ab} \partial_{cd]} \equiv 0. \quad (2.5)$$

Acting on arbitrary functions,  $\Phi$ ,  $\Phi'$ , and their products, the section condition leads to

$$\partial_{[ab} \partial_{cd]} \Phi = \partial_{[ab} \partial_{c]d} \Phi = 0 \quad (\text{weak constraint}), \quad (2.6)$$

$$\partial_{[ab} \Phi \partial_{cd]} \Phi' = \frac{1}{2} \partial_{[ab} \Phi \partial_{c]d} \Phi' - \frac{1}{2} \partial_{d[a} \Phi \partial_{b]c} \Phi' = 0 \quad (\text{strong constraint}). \quad (2.7)$$

**Diffeomorphism.** Diffeomorphism symmetry in  $\mathbf{SL}(N)$  U-gravity is generated by a generalized Lie derivative,

$$\begin{aligned} \hat{\mathcal{L}}_X T^{a_1 a_2 \dots a_p}_{b_1 b_2 \dots b_q} := & \frac{1}{2} X^{cd} \partial_{cd} T^{a_1 a_2 \dots a_p}_{b_1 b_2 \dots b_q} + \frac{1}{2} \left( \frac{1}{2} p - \frac{1}{2} q + \omega \right) \partial_{cd} X^{cd} T^{a_1 a_2 \dots a_p}_{b_1 b_2 \dots b_q} \\ & - \sum_{i=1}^p T^{a_1 \dots c \dots a_p}_{b_1 b_2 \dots b_q} \partial_{cd} X^{a_i d} + \sum_{j=1}^q \partial_{b_j d} X^{cd} T^{a_1 a_2 \dots a_p}_{b_1 \dots c \dots b_q}. \end{aligned} \quad (2.8)$$

Here we let the tensor density,  $T^{a_1 a_2 \dots a_p}_{b_1 b_2 \dots b_q}$ , carry the ‘total’ weight,  $\frac{1}{2}p - \frac{1}{2}q + \omega$ , such that each upper or lower index contributes to the total weight by  $+\frac{1}{2}$  or  $-\frac{1}{2}$  respectively, while  $\omega$  corresponds to a possible ‘extra’ weight.

In particular, the generalized Lie derivative of the Kronecker delta symbol is trivial,

$$\hat{\mathcal{L}}_X \delta^a_b = 0, \quad (2.9)$$

and the commutator of the generalized Lie derivatives is closed by a generalized bracket [25],

$$[\hat{\mathcal{L}}_X, \hat{\mathcal{L}}_Y] = \hat{\mathcal{L}}_{[X, Y]_G}, \quad [X, Y]_G^{ab} = \frac{1}{2} X^{cd} \partial_{cd} Y^{ab} - \frac{3}{2} X^{[ab} \partial_{cd} Y^{cd]} - (X \leftrightarrow Y). \quad (2.10)$$

It is a somewhat surprising result of us that the above definition of the generalized Lie derivative — including the total weight — is independent of the rank of the duality group, or  $N$ , and thus is identical to the known one in [25, 79] for the case of  $N = 5$ .

**U-metric.** The only geometric object in  $\mathbf{SL}(N)$  U-gravity is a metric, or *U-metric*, which is a generic non-degenerate  $N \times N$  symmetric matrix, obeying surely the section condition,

$$M_{ab} = M_{ba} = M_{(ab)}. \quad (2.11)$$

Like in Riemannian geometry, the U-metric with its inverse,  $M^{ab}$ , may freely lower or raise the positions of the  $N$ -dimensional  $\mathbf{SL}(N)$  vector indices.

**Integral measure.** While the U-metric has no extra weight, its determinant,  $M \equiv \det(M_{ab})$ , acquires an extra weight,  $\omega = 4 - N$ . The duality invariant integral measure is then

$$|M|^{\frac{1}{4-N}}. \quad (2.12)$$

**Semi-covariant derivative and semi-covariant Riemann curvature.** We define a semi-covariant derivative,

$$\begin{aligned} \nabla_{cd} T^{a_1 a_2 \dots a_p}_{b_1 b_2 \dots b_q} &:= \partial_{cd} T^{a_1 a_2 \dots a_p}_{b_1 b_2 \dots b_q} + \frac{1}{2} \left( \frac{1}{2}p - \frac{1}{2}q + \omega \right) \Gamma_{cde} T^{a_1 a_2 \dots a_p}_{b_1 b_2 \dots b_q} \\ &\quad - \sum_{i=1}^p T^{a_1 \dots e \dots a_p}_{b_1 b_2 \dots b_q} \Gamma_{cde}^{ai} + \sum_{j=1}^q \Gamma_{cdb_j} T^{a_1 a_2 \dots a_p}_{b_1 \dots e \dots b_q}, \end{aligned} \quad (2.13)$$

and a semi-covariant Riemann curvature,

$$\begin{aligned} S_{abcd} &:= 3\partial_{[ab} \Gamma_{e][cd]}^e + 3\partial_{[cd} \Gamma_{e][ab]}^e + \frac{1}{4} \Gamma_{abe}^e \Gamma_{cdf}^f + \frac{1}{2} \Gamma_{abe}^f \Gamma_{cdf}^e \\ &\quad + \Gamma_{ab[c}^e \Gamma_{d]ef}^f + \Gamma_{cd[a}^e \Gamma_{b]ef}^f + \Gamma_{ea[c}^f \Gamma_{d]fb}^e - \Gamma_{eb[c}^f \Gamma_{d]fa}^e. \end{aligned} \quad (2.14)$$

The semi-covariant derivative obeys the Leibniz rule and annihilates the Kronecker delta symbol,

$$\nabla_{cd} \delta^a_b = 0. \quad (2.15)$$

A crucial defining property of the semi-covariant Riemann curvature is that, under arbitrary transformation of the connection it transforms as *total derivative*,

$$\delta S_{abcd} = 3\nabla_{[ab} \delta \Gamma_{e][cd]}^e + 3\nabla_{[cd} \delta \Gamma_{e][ab]}^e. \quad (2.16)$$

Further, the semi-covariant Riemann curvature satisfies precisely the same symmetric properties as the ordinary Riemann curvature, including the Bianchi identity,

$$S_{abcd} = S_{[ab][cd]} = S_{cdab}, \quad S_{[abc]d} = 0. \quad (2.17)$$

**Connection.** The connection of the semi-covariant derivative and the semi-covariant Riemann curvature is given by

$$\begin{aligned} \Gamma_{abcd} &= A_{abcd} + \frac{1}{2}(A_{acbd} - A_{adbc} + A_{bdac} - A_{bcad}) \\ &+ \frac{1}{N-2}(M_{ac}A^e_{(bd)e} - M_{ad}A^e_{(bc)e} + M_{bd}A^e_{(ac)e} - M_{bc}A^e_{(ad)e}), \end{aligned} \quad (2.18)$$

where we set

$$A_{abcd} := -\frac{1}{2}\partial_{ab}M_{cd} + \frac{1}{2(N-4)}M_{cd}\partial_{ab}\ln|M|. \quad (2.19)$$

This connection is *the unique solution* to the following five constraints:<sup>4</sup>

$$\Gamma_{abcd} + \Gamma_{abdc} = 2A_{abcd}, \quad (2.20)$$

$$\Gamma_{abc}{}^d + \Gamma_{bac}{}^d = 0, \quad (2.21)$$

$$\Gamma_{abc}{}^d + \Gamma_{bca}{}^d + \Gamma_{cab}{}^d = 0, \quad (2.22)$$

$$\Gamma_{cab}{}^c + \Gamma_{cba}{}^c = 0, \quad (2.23)$$

$$\mathcal{P}_{abcd}{}^{efgh}\Gamma_{efgh} = 0. \quad (2.24)$$

The first relation (2.20) is equivalent to the U-metric compatibility condition,

$$\nabla_{ab}M_{cd} = 0. \quad (2.25)$$

The second condition (2.21) is natural from the skew-symmetric nature of the coordinates,  $x^{(ab)} = 0$  and hence  $\partial_{(ab)} = \nabla_{(ab)} = 0$ . The next two constraints, (2.22) and (2.23), make the semi-covariant derivative compatible with the generalized Lie derivative as well as with the generalized bracket,

$$\hat{\mathcal{L}}_X(\partial) = \hat{\mathcal{L}}_X(\nabla), \quad [X, Y]_G(\partial) = [X, Y]_G(\nabla). \quad (2.26)$$

The last formula (2.24) is a projection condition which we impose intentionally in order to ensure the uniqueness.

**Projection operator.** The eight-index projection operator, used in (2.24), is explicitly,

$$\begin{aligned} \mathcal{P}_{abcd}{}^{klmn} &= \frac{1}{2}\delta_{[a}^{[k}\delta_{b]}^{l]}\delta_{[c}^{[m}\delta_{d]}^{n]} + \frac{1}{2}\delta_{[c}^{[k}\delta_{d]}^{l]}\delta_{[a}^{[m}\delta_{b]}^{n]} + \frac{1}{2}M_{c[a}\delta_{b]}^m M^{n[k}\delta_d^{l]} - \frac{1}{2}M_{c[a}\delta_{b]}^{[k} M^{l]n}\delta_d^m \\ &+ \frac{1}{N-2}\left(\delta_{[a}^n M_{b][c} M^{m[k}\delta_{d]}^{l]} + \delta_{[c}^n M_{d][a} M^{m[k}\delta_{b]}^{l]} - M_{c[a} M_{b]d} M^{m[k} M^{l]n}\right), \end{aligned} \quad (2.27)$$

which satisfies

$$\begin{aligned} \mathcal{P}_{abcd}{}^{pqrs}\mathcal{P}_{pqrs}{}^{klmn} &= \mathcal{P}_{abcd}{}^{klmn}, & \mathcal{P}_{abs}{}^{sklmn} &= 0, \\ \mathcal{P}_{abcd}{}^{klmn} &= \mathcal{P}_{[ab]cd}{}^{[kl]mn}, & \mathcal{P}_{ab[cd]}{}^{klmn} &= \mathcal{P}_{cd[ab]}{}^{klmn}. \end{aligned} \quad (2.28)$$

<sup>4</sup>See [65, 66] for the analogous constraints in DFT.

Crucially, the projection operator dictates the anomalous terms in the diffeomorphic transformations of the semi-covariant derivative and the semi-covariant Riemann curvature,

$$\begin{aligned} (\delta_X - \hat{\mathcal{L}}_X) (\nabla_{ab} T^{c_1 \dots c_p}_{d_1 d_2 \dots d_q}) &= - \sum_{i=1}^p T^{c_1 \dots e \dots c_p}_{d_1 \dots d_q} \Omega_{abe}{}^{c_i} + \sum_{j=1}^q \Omega_{abd_j}{}^e T^{c_1 c_2 \dots c_p}_{d_1 \dots e \dots d_q}, \\ (\delta_X - \hat{\mathcal{L}}_X) S_{abcd} &= 2 \nabla_{e[a} \Omega_{b][cd]}{}^e + 2 \nabla_{e[c} \Omega_{d][ab]}{}^e, \\ \Omega_{abcd} &= \mathcal{P}_{abcd}{}^{klm}{}_n \partial_{kl} \partial_{me} X^{ne}. \end{aligned} \quad (2.29)$$

**Complete covariantizations.** Both the semi-covariant derivative and the semi-covariant Riemann curvature can be fully covariantized by (anti-)symmetrizing or contracting the  $\mathbf{SL}(N)$  vector indices properly [30],

$$\begin{aligned} \nabla_{[ab} T_{c_1 c_2 \dots c_q]}, \quad \nabla_{ab} T^a, \quad \nabla^a{}_b T_{[ca]} + \nabla^a{}_c T_{[ba]}, \quad \nabla^a{}_b T_{(ca)} - \nabla^a{}_c T_{(ba)}, \\ \nabla_{ab} T^{[abc_1 c_2 \dots c_q]} \quad (\text{divergence}), \quad \nabla_{ab} \nabla^{[ab} T^{c_1 c_2 \dots c_q]} \quad (\text{Laplacian}), \end{aligned} \quad (2.30)$$

and

$$\begin{aligned} S_{ab} &:= S_{acb}{}^c = S_{ba} \quad (\text{Ricci curvature}), \\ S &:= M^{ab} S_{ab} = S_{ab}{}^{ab} \quad (\text{scalar curvature}). \end{aligned} \quad (2.31)$$

**Action.** The action of  $\mathbf{SL}(N)$  U-gravity is given by the fully covariant scalar curvature,

$$\int_{\Sigma} M^{\frac{1}{4-N}} S, \quad (2.32)$$

where the integral is taken over a section,  $\Sigma$ .

**The Einstein equation of motion.** The equation of motion corresponds to the vanishing of the ‘Einstein’ tensor,

$$S_{ab} + \frac{1}{2(N-4)} M_{ab} S = 0. \quad (2.33)$$

Diffeomorphism symmetry of the action implies a conservation relation,

$$\nabla^c{}_{[a} S_{b]c} + \frac{3}{8} \nabla_{ab} S = 0. \quad (2.34)$$

**Two inequivalent sections.** Up to  $\mathbf{SL}(N)$  duality rotations, there exist two inequivalent solutions to the section condition, which we denote here by  $\Sigma_{N-1}$  and  $\Sigma_3$ .

1.  $\Sigma_{N-1}$  is an  $(N-1)$ -dimensional section given by

$$\partial_{\alpha\beta} = 0, \quad \partial_{\alpha N} \neq 0, \quad (2.35)$$

where  $\alpha, \beta = 1, 2, \dots, N-1$ .

2.  $\Sigma_3$  is a three-dimensional section characterized by

$$\partial_{\mu i} = 0, \quad \partial_{ij} = 0, \quad \partial_{\mu\nu} \neq 0, \quad (2.36)$$

where  $\mu, \nu = 1, 2, 3$  and  $i, j = 4, 5, \dots, N$ . In this case, we may dualize the nontrivial three coordinates, using a three-dimensional Levi-Civita symbol,  $\varepsilon_{123} \equiv 1$ ,

$$\tilde{x}_\mu \equiv \frac{1}{2} \varepsilon_{\mu\nu\rho} x^{\nu\rho}, \quad \tilde{\partial}^\mu \tilde{x}_\nu = \delta^\mu{}_\nu. \quad (2.37)$$



For a triplet of arbitrary functions, we note [80]

$$\partial_{[ab}\Phi\partial_{c][d}\Phi'\partial_{ef]}\Phi'' = 0 \quad \text{on} \quad \Sigma_{N-1}, \quad \partial_{[ab}\Phi\partial_{c][d}\Phi'\partial_{ef]}\Phi'' \neq 0 \quad \text{on} \quad \Sigma_3. \quad (2.38)$$

Since this is an  $\mathbf{SL}(N)$  covariant statement, the two sections,  $\Sigma_{N-1}$  and  $\Sigma_3$ , are duality inequivalent. More than one solution to a section condition has been also reported in EFT [37, 38].

### Riemannian reductions.

1. To perform the Riemannian reduction to  $\Sigma_{N-1}$  (2.35), we parametrize the U-metric in terms of  $(N-1)$ -dimensional Riemannian metric,  $g_{\alpha\beta}$ , a vector,  $v^\alpha$ , and a scalar,  $\phi$  [30],

$$M_{ab} = \begin{pmatrix} \frac{g_{\alpha\beta}}{\sqrt{|g|}} & v_\alpha \\ v_\beta & \sqrt{|g|}(-e^\phi + v^2) \end{pmatrix}, \quad |M|^{\frac{1}{4-N}} = e^{\frac{1}{4-N}\phi} \sqrt{|g|}. \quad (2.39)$$

The U-gravity scalar curvature (2.31) reduces upon the section,  $\Sigma_{N-1}$ , to

$$S|_{\Sigma_{N-1}} = 2e^{-\phi} \left[ R_g - \frac{(N-3)(3N-8)}{4(N-4)^2} \partial_\alpha \phi \partial^\alpha \phi + \frac{N-2}{N-4} \Delta \phi + \frac{1}{2} e^{-\phi} (\nabla_\alpha v^\alpha)^2 \right]. \quad (2.40)$$

The vector field can be dualized to an  $(N-2)$ -form potential.

2. For the Riemannian reduction to  $\Sigma_3$  (2.36), we parametrize (the inverse of) the U-metric, employing ‘dual’ upside-down notations [80],

$$M^{ab} = \begin{pmatrix} \frac{\tilde{g}^{\mu\nu}}{\sqrt{|\tilde{g}|}} & -\tilde{v}^{j\mu} \\ -\tilde{v}^{i\nu} & \sqrt{|\tilde{g}|} \left( e^{-\tilde{\phi}} \tilde{\mathcal{M}}^{ij} + \tilde{v}^{i\lambda} \tilde{v}^j{}_\lambda \right) \end{pmatrix}, \quad |M|^{\frac{1}{4-N}} = e^{\frac{N-3}{4-N}\tilde{\phi}} \sqrt{|\tilde{g}|}. \quad (2.41)$$

The U-gravity scalar curvature (2.31) reduces upon the section,  $\Sigma_3$ , to

$$S|_{\Sigma_3} = -2R_{\tilde{g}} + \frac{(N-3)(3N-8)}{2(N-4)^2} \tilde{\partial}^\mu \tilde{\phi} \tilde{\partial}_\mu \tilde{\phi} - \frac{4(N-3)}{N-4} \tilde{\Delta} \tilde{\phi} - \frac{1}{2} \tilde{\partial}^\mu \tilde{\mathcal{M}}_{ij} \tilde{\partial}_\mu \tilde{\mathcal{M}}^{ij} + e^{\tilde{\phi}} \tilde{\mathcal{M}}_{ij} \tilde{\nabla}^\mu \tilde{v}^i{}_\mu \tilde{\nabla}^\nu \tilde{v}^j{}_\nu, \quad (2.42)$$

which manifests  $\mathbf{SL}(N-3)$  *S-duality*.

**Non-Riemannian backgrounds.** When the upper left  $(N-1) \times (N-1)$  block of the U-metric is degenerate — where  $\frac{g_{\alpha\beta}}{\sqrt{|g|}}$  is positioned in (2.39) — the Riemannian metric ceases to exist upon  $\Sigma_{N-1}$ . Nevertheless,  $\mathbf{SL}(N)$  U-gravity has no problem with describing such a non-Riemannian background, as long as the whole  $N \times N$  U-metric is non-degenerate. Similarly upon  $\Sigma_3$ , U-gravity may allow the upper left  $3 \times 3$  block of the inverse of the U-metric (2.41) to be degenerate (see section 3.8 for further discussion with examples).<sup>5</sup>

### 3 Exposition

In this section we provide detailed exposition of the main results listed in section 2. All the mathematical analyses are parallel to those in the DFT-geometry of [57, 61, 65, 66, 81].

<sup>5</sup>Consult also [61] for a parallel discussion in DFT.

### 3.1 Equivalence between the coordinate gauge symmetry and the section condition

Here, following a parallel argument in DFT [61], we show the equivalence between the coordinate gauge symmetry invariance (2.4),

$$\Phi(x + s\Delta) = \Phi(x), \quad \Delta^{ab} = \frac{1}{(N-4)!} \epsilon^{abc_1 \dots c_{N-4} de} \phi_{c_1 \dots c_{N-4}} \partial_{de} \varphi, \quad (3.1)$$

and the section condition (2.7),

$$\partial_{[ab} \Phi \partial_{cd]} \Phi' = \frac{1}{2} \partial_{[ab} \Phi \partial_{c]d} \Phi' - \frac{1}{2} \partial_{d[a} \Phi \partial_{b]c} \Phi' = 0 \quad (\text{strong constraint}), \quad (3.2)$$

$$\partial_{[ab} \partial_{cd]} \Phi = \partial_{[ab} \partial_{c]d} \Phi = 0 \quad (\text{weak constraint}). \quad (3.3)$$

Note that, in (3.1) we put a continuous real parameter,  $s$ , in order to control the shift.

First of all, from the standard series expansion of  $\Phi(x + s\Delta)$  in  $s$ , it is clear that the strong constraint, (3.2), implies the invariance (3.1). The converse is also true: taking derivative at  $s = 0$ , we get

$$0 = \left. \frac{d}{ds} \Phi(x + s\Delta) \right|_{s=0} = \frac{1}{2} \Delta^{ab} \partial_{ab} \Phi = \frac{1}{2(N-4)!} \epsilon^{c_1 \dots c_{N-4} deab} \phi_{c_1 \dots c_{N-4}} \partial_{de} \varphi \partial_{ab} \Phi. \quad (3.4)$$

This shows that the invariance (3.1) indeed implies the strong constraint (3.2). Further, from the strong constraint, it follows that the following  $\frac{N(N-1)}{2} \times \frac{N(N-1)}{2}$  matrix is nilpotent,

$$\mathcal{K}^{ab}{}_{cd} = \frac{1}{(N-4)!} \epsilon^{abe_1 \dots e_{N-4} fg} \phi_{e_1 \dots e_{N-4}} \partial_{fg} \partial_{cd} \varphi, \quad \frac{1}{2} \mathcal{K}^{ab}{}_{cd} \mathcal{K}^{cd}{}_{ef} = 0. \quad (3.5)$$

Since any nilpotent matrix is traceless,<sup>6</sup> we have

$$\mathcal{K}^{ab}{}_{ab} = \frac{1}{(N-4)!} \epsilon^{e_1 \dots e_{N-4} abfg} \phi_{e_1 \dots e_{N-4}} \partial_{ab} \partial_{fg} \varphi = 0, \quad (3.6)$$

which leads to the weak constraint (3.3),

$$\partial_{[ab} \partial_{cd]} \varphi = 0. \quad (3.7)$$

In this way, the strong constraint (3.2) implies the weak constraint (3.3), and is actually equivalent to the coordinate gauge symmetry invariance (3.1). This completes our proof.

### 3.2 Projection operator

The eight-index projection operator (2.27),

$$\begin{aligned} \mathcal{P}_{abcd}{}^{klmn} = & \frac{1}{2} \delta_{[a}^{[k} \delta_{b]}^{l]} \delta_{[c}^{[m} \delta_{d]}^{n]} + \frac{1}{2} \delta_{[c}^{[k} \delta_{d]}^{l]} \delta_{[a}^{[m} \delta_{b]}^{n]} + \frac{1}{2} M_{c[a} \delta_{b]}^m M^{n[k} \delta_d^{l]} - \frac{1}{2} M_{c[a} \delta_{b]}^{[k} M^{l]n} \delta_d^m \\ & + \frac{1}{N-2} \left( \delta_{[a}^n M_{b]}^{[c} M^{m[k} \delta_d^{l]} + \delta_{[c}^n M_{d]}^{[a} M^{m[k} \delta_b^{l]} - M_{c[a} M_{b]d} M^{m[k} M^{l]n} \right), \end{aligned} \quad (3.8)$$

<sup>6</sup>All the diagonal elements of the Jordan normal form of a nilpotent matrix are trivial [61].

satisfies the ‘projection’ property,

$$\mathcal{P}_{abcd}{}^{pqrs}\mathcal{P}_{pqrs}{}^{klmn} = \mathcal{P}_{abcd}{}^{klmn}. \quad (3.9)$$

The verification of this identity requires straightforward yet tedious computations, which can be simplified by noting symmetric properties,

$$\mathcal{P}_{abcd}{}^{klmn} = \mathcal{P}_{[ab]cd}{}^{[kl]mn}, \quad \mathcal{P}_{ab[cd]}{}^{klmn} = \mathcal{P}_{cd[ab]}{}^{klmn}, \quad (3.10)$$

and ‘trace’ properties,

$$\begin{aligned} \mathcal{P}^s{}_{asb}{}^{klmn} &= \frac{1}{2}(N-2)\delta_{(a}{}^m M^{n[k}\delta_{b)}^{l]} - \frac{N}{2(N-2)}M_{ab}M^{m[k}M^{l]n}, \\ \mathcal{P}^s{}_{abs}{}^{klmn} &= \delta_{(a}{}^m M^{n[k}\delta_{b)}^{l]} + \frac{N-1}{N-2}M_{ab}M^{m[k}M^{l]n}, \\ \mathcal{P}^{rs}{}_{rs}{}^{klmn} &= -\left(\frac{N^2-2N+2}{N-2}\right)M^{m[k}M^{l]n}, \\ \mathcal{P}_{abs}{}^{sklmn} &= 0. \end{aligned} \quad (3.11)$$

The traces are related to each other by

$$\mathcal{P}^s{}_{asb}{}^{klmn} = \frac{1}{2}(N-2)\mathcal{P}^s{}_{abs}{}^{klmn} + \frac{1}{2}M_{ab}\mathcal{P}^{rs}{}_{rs}{}^{klmn}. \quad (3.12)$$

It is also useful to note

$$\mathcal{P}_{[abc]d}{}^{klmn} = \mathcal{P}_{[abcd]}{}^{[klmn]} = \delta_{[a}^{[k}\delta_b^l\delta_c^m\delta_d^{n]}. \quad (3.13)$$

As we shall see below, the projection operator plays crucial roles in U-gravity.<sup>7</sup> Compared to the ordinary Riemannian geometry, the existence of a projection operator and its key role appear to be novel distinct features of the extended-yet-gauged spacetime geometries, such as DFT-geometry in [65–71] and the present  $\mathbf{SL}(N)$  U-gravity.

### 3.3 Compatibility of the semi-covariant derivative

Here we discuss the compatibilities of the semi-covariant derivative, firstly with the generalized Lie derivative, secondly with the generalized bracket, and lastly with the U-metric.

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<sup>7</sup>The construction of the projection operator (3.8) is one of the major improvements made in this paper compared to the previous work on  $\mathbf{SL}(5)$  U-geometry [30]. An operator therein, called  $J_{abcd}{}^{klmn}$ , is consistently related to the projection operator by

$$\begin{aligned} J_{abcd}{}^{klmn} &= \frac{1}{2}\delta_{[a}^{[k}\delta_{b]}^{l]}\delta_{[c}^{[m}\delta_{d]}^{n]} + \frac{1}{2}\delta_{[c}^{[k}\delta_{d]}^{l]}\delta_{[a}^{[m}\delta_{b]}^{n]} + \frac{1}{N-2}\left(\delta_{[a}^n M_{b][c}M^{m[k}\delta_{d]}^{l]} + \delta_{[c}^n M_{d][a}M^{m[k}\delta_{b]}^{l]}\right) \\ &= \mathcal{P}_{abcd}{}^{klmn} - M_{c[a}\mathcal{P}^s{}_{b]ds}{}^{klmn} - \frac{N}{N^2-2N+2}M_{c[a}M_{b]d}\mathcal{P}^{rs}{}_{rs}{}^{klmn}. \end{aligned}$$

Specifically, we start by postulating the generalized Lie derivative and the semi-covariant derivative to take the following forms,

$$\begin{aligned}
 \hat{\mathcal{L}}_X T^{a_1 a_2 \dots a_p}_{b_1 b_2 \dots b_q} &:= \frac{1}{2} X^{cd} \partial_{cd} T^{a_1 a_2 \dots a_p}_{b_1 b_2 \dots b_q} + \alpha(p, q, \omega) \partial_{cd} X^{cd} T^{a_1 a_2 \dots a_p}_{b_1 b_2 \dots b_q} \\
 &\quad - \sum_{i=1}^p T^{a_1 \dots c \dots a_p}_{b_1 b_2 \dots b_q} \partial_{cd} X^{a_i d} + \sum_{j=1}^q \partial_{b_j d} X^{cd} T^{a_1 a_2 \dots a_p}_{b_1 \dots c \dots b_q}, \\
 \nabla_{cd} T^{a_1 a_2 \dots a_p}_{b_1 b_2 \dots b_q} &:= \partial_{cd} T^{a_1 a_2 \dots a_p}_{b_1 b_2 \dots b_q} + \bar{\alpha}(p, q, \omega) \Gamma_{cde}{}^e T^{a_1 a_2 \dots a_p}_{b_1 b_2 \dots b_q} \\
 &\quad - \sum_{i=1}^p T^{a_1 \dots e \dots a_p}_{b_1 b_2 \dots b_q} \Gamma_{cde}{}^{a_i} + \sum_{j=1}^q \Gamma_{cdb_j}{}^e T^{a_1 a_2 \dots a_p}_{b_1 \dots e \dots b_q}. \quad (3.14)
 \end{aligned}$$

Here,  $\alpha(p, q, \omega)$  and  $\bar{\alpha}(p, q, \omega)$  are yet-undetermined total weights which may depend on  $p, q, \omega$ , i.e. the numbers of upper, lower indices and the extra weight. Below, in section 3.3.1, by demanding the compatibility with the generalized Lie derivative, we shall fix the dependency and derive the final expression,

$$\alpha(p, q, \omega) = \bar{\alpha}(p, q, \omega) = \frac{1}{2} \left( \frac{1}{2} p - \frac{1}{2} q + \omega \right), \quad (3.15)$$

which is linear in  $p, q, \omega$  and remarkably independent of  $N$ . This result will, in particular, ensure that both the generalized Lie derivative and the semi-covariant derivative annihilate the Kronecker delta symbol,

$$\hat{\mathcal{L}}_X \delta_b^a = 0, \quad \nabla_{cd} \delta_b^a = 0. \quad (3.16)$$

Further, while the extra weight of the U-metric is trivial, its determinant,  $M \equiv \det(M_{ab})$ , acquires an extra weight,  $\omega = 4 - N$ , since under diffeomorphism, it transforms as

$$\delta_X M = \frac{1}{2} X^{ab} \partial_{ab} M + \frac{1}{2} (4 - N) \partial_{ab} X^{ab} M. \quad (3.17)$$

This implies that *the duality invariant integral measure* with unit extra weight is

$$|M|^{\frac{1}{4-N}}. \quad (3.18)$$

It is instructive to note that, irrespective of the choice of  $\alpha(p, q, \omega)$ , upon the section condition, the commutator of the generalized Lie derivative is closed by a generalized bracket [25],

$$\left[ \hat{\mathcal{L}}_X, \hat{\mathcal{L}}_Y \right] T^{a_1 a_2 \dots a_p}_{b_1 b_2 \dots b_q} = \hat{\mathcal{L}}_{[X, Y]_G} T^{a_1 a_2 \dots a_p}_{b_1 b_2 \dots b_q}, \quad (3.19)$$

$$[X, Y]_G^{ab} = \frac{1}{2} X^{cd} \partial_{cd} Y^{ab} - \frac{3}{2} X^{[ab} \partial_{cd} Y^{cd]} - \frac{1}{2} Y^{cd} \partial_{cd} X^{ab} + \frac{3}{2} Y^{[ab} \partial_{cd} X^{cd]}. \quad (3.20)$$

Further, it is obvious from this expression that the generalized bracket satisfies up to the section condition,

$$\begin{aligned}
 [X, Y]_G^{ab} \partial_{ab} \Phi &= \frac{1}{2} \left( X^{cd} \partial_{cd} Y^{ab} - Y^{cd} \partial_{cd} X^{ab} \right) \partial_{ab} \Phi, \\
 \partial_{ab} \left( [X, Y]_G^{ab} \right) &= \frac{1}{2} \left( X^{cd} \partial_{cd} \partial_{ab} Y^{ab} - Y^{cd} \partial_{cd} \partial_{ab} X^{ab} \right). \quad (3.21)
 \end{aligned}$$

### 3.3.1 Compatibility with the generalized Lie derivative

If we replace all the ordinary derivatives by semi-covariant derivatives in the definition of the generalized Lie derivative expressed in (3.14), we get

$$\begin{aligned} \left[ \hat{\mathcal{L}}_X(\nabla) - \hat{\mathcal{L}}_X(\partial) \right] T^{a_1 \dots a_p}_{b_1 \dots b_q} &= X^{cd} \left[ \left( \frac{1}{2} \bar{\alpha} + \alpha \bar{\beta} \right) \Gamma_{cde}{}^e + 2\alpha \Gamma_{e[cd]}{}^e \right] T^{a_1 \dots a_p}_{b_1 \dots b_q} \\ &\quad - \sum_{i=1}^p T^{a_1 \dots e \dots a_p}_{b_1 \dots b_q} \left[ \frac{3}{2} X^{cd} \Gamma_{[cde]}{}^{a_i} + X^{a_i d} (\bar{\beta} \Gamma_{edc}{}^c - \Gamma_{ecd}{}^c) \right] \\ &\quad + \sum_{j=1}^q \left[ \frac{3}{2} X^{cd} \Gamma_{[cdb_j]}{}^e + X^{ed} (\bar{\beta} \Gamma_{b_j dc}{}^c - \Gamma_{b_j cd}{}^c) \right] T^{a_1 \dots a_p}_{b_1 \dots e \dots b_q}, \end{aligned} \quad (3.22)$$

where we set for the parameter,  $X^{ab}$ ,

$$\bar{\beta} \equiv \bar{\alpha}(2, 0, 0). \quad (3.23)$$

The compatibility of the semi-covariant derivative with the generalized Lie derivative means that the right hand side of (3.22) should vanish algebraically. In order to achieve this, it is required that the four-index quantity,  $\Gamma_{[abc]}{}^d$ , should be, at least, related to the two-index quantities,  $\Gamma_{eab}{}^e$  and  $\Gamma_{abe}{}^e$ . There is one unique such an ansatz which is self-consistent,<sup>8</sup>

$$\Gamma_{[abc]}{}^d = \frac{1}{N-2} \hat{\Gamma}_{[ab} \delta_c]{}^d, \quad \hat{\Gamma}_{ab} = 3\Gamma_{[abe]}{}^e. \quad (3.24)$$

Note that the left and right hand sides of this ansatz share the same anti-symmetric properties, and also that the contractions of the two indices, one lower and the other upper (for example  $c$  and  $d$ ), agree.

Assuming the ansatz (3.24), the expression (3.22) reduces to

$$\begin{aligned} &\left[ \hat{\mathcal{L}}_X(\nabla) - \hat{\mathcal{L}}_X(\partial) \right] T^{a_1 \dots a_p}_{b_1 \dots b_q} \\ &= X^{cd} \left[ \left( \frac{1}{2} \bar{\alpha} + \alpha \bar{\beta} + \frac{q-p}{2(N-2)} \right) \Gamma_{cde}{}^e + \left( 2\alpha + \frac{q-p}{N-2} \right) \Gamma_{e[cd]}{}^e \right] T^{a_1 \dots a_p}_{b_1 \dots b_q} \\ &\quad - \sum_{i=1}^p X^{a_i d} \left[ \left( \bar{\beta} - \frac{1}{N-2} \right) \Gamma_{edc}{}^c + \frac{N-4}{N-2} \Gamma_{e[cd]}{}^c + \Gamma_{c(ed)}{}^c \right] T^{a_1 \dots e \dots a_p}_{b_1 \dots b_q} \\ &\quad + \sum_{j=1}^q X^{ed} \left[ \left( \bar{\beta} - \frac{1}{N-2} \right) \Gamma_{b_j dc}{}^c + \frac{N-4}{N-2} \Gamma_{c[b_j d]}{}^c + \Gamma_{c(b_j d)}{}^c \right] T^{a_1 \dots a_p}_{b_1 \dots e \dots b_q}. \end{aligned} \quad (3.25)$$

Now, each line above should vanish separately. More precisely, with the skew-symmetry,  $\Gamma_{abc}{}^d = \Gamma_{[ab]c}{}^d$ , we should require

$$\left( \frac{1}{2} \bar{\alpha} + \alpha \bar{\beta} + \frac{q-p}{2(N-2)} \right) \Gamma_{abc}{}^c + \left( 2\alpha + \frac{q-p}{N-2} \right) \Gamma_{c[ab]}{}^c = 0, \quad (3.26)$$

$$\left( \bar{\beta} - \frac{1}{N-2} \right) \Gamma_{abc}{}^c + \frac{N-4}{N-2} \Gamma_{c[ab]}{}^c = 0, \quad (3.27)$$

$$\Gamma_{c(ab)}{}^c = 0. \quad (3.28)$$

<sup>8</sup>The division by  $N-2$  in (3.24) needs not cause any alarm to exclude the case of  $N=2$ , since after all we shall have  $\Gamma_{[abc]}{}^d = 0$  (2.22).

Equation (3.27) gives an expression,  $\Gamma_{c[ab]}^c = \frac{1-(N-2)\bar{\beta}}{N-4}\Gamma_{abc}^c$ . Substituting this into (3.26), we get

$$\left[ \frac{1}{2}\bar{\alpha} + \alpha\bar{\beta} + \frac{q-p}{2(N-2)} + \left( 2\alpha + \frac{q-p}{N-2} \right) \frac{1-(N-2)\bar{\beta}}{N-4} \right] \Gamma_{abc}^c = 0. \quad (3.29)$$

There is a good reason for the contraction,  $\Gamma_{abc}^c$ , to be nontrivial: as we shall discuss more in section 3.3.3, the compatibility of the semi-covariant derivative with the U-metric, and hence with its determinant, implies for some value<sup>9</sup> of  $\omega^*$ ,

$$\nabla_{ab}M = \partial_{ab}M + \bar{\alpha}(0,0,\omega^*)\Gamma_{abc}^cM \equiv 0. \quad (3.30)$$

For this to hold,  $\Gamma_{abc}^c$  should not vanish in general. Thus, eq. (3.29) tells us

$$\frac{1}{2}\bar{\alpha} + \alpha\bar{\beta} + \frac{q-p}{2(N-2)} + \left( 2\alpha + \frac{q-p}{N-2} \right) \frac{1-(N-2)\bar{\beta}}{N-4} = 0. \quad (3.31)$$

In particular, for the special case of  $p = 2, q = 0, \omega = 0$ , this reduces to

$$(N\bar{\beta} - 2)(2\beta - 1) = 0, \quad (3.32)$$

where, like (3.23), we set  $\beta \equiv \alpha(2,0,0)$ . Hence, we have either  $\bar{\beta} = \frac{2}{N}$  or  $\beta = \frac{1}{2}$ . If  $\bar{\beta} = \frac{2}{N}$ , eq. (3.31) would get simplified to give  $\bar{\alpha}(p,q,\omega) = \frac{p-q}{N}$ . However, this is not a desired result. In order to meet the compatibility with the U-metric determinant (3.30),  $\bar{\alpha}(p,q,\omega)$  must depend nontrivially on  $\omega$  rather than being independent of it. Therefore, we should choose  $\beta = \frac{1}{2}$ .

Now, rather than trying to look for the most general solution, for simplicity we focus on the case of  $\beta = \bar{\beta} = \frac{1}{2}$  and search for a ‘linear’ solution. Then, eq. (3.31) implies a more generic equality,  $\alpha = \bar{\alpha}$ , and naturally we are lead to the final expression for the total weight, i.e. (3.15). Further, (3.26) and (3.27) reduce to

$$\Gamma_{abc}^c + 2\Gamma_{c[ab]}^c = \hat{\Gamma}_{ab} = 0, \quad (3.33)$$

and thus, from (3.24) and (3.28), we arrive at the conclusion: the conditions for the compatibility of the semi-covariant derivative with the generalized Lie derivative are

$$\Gamma_{[abc]}^d = 0, \quad \Gamma_{c(ab)}^c = 0, \quad \alpha(p,q,\omega) = \bar{\alpha}(p,q,\omega) = \frac{1}{2} \left( \frac{1}{2}p - \frac{1}{2}q + \omega \right). \quad (3.34)$$

### 3.3.2 Compatibility with the generalized bracket

If we replace all the ordinary derivatives by semi-covariant derivatives in the definition of the generalized bracket (3.20), we get, in a similar fashion to (3.22),

$$\begin{aligned} [X, Y]_G^{ab}(\nabla) - [X, Y]_G^{ab}(\partial) &= \frac{1}{2} \left( Y^{ab}X^{cd} - X^{ab}Y^{cd} \right) (\bar{\beta}\Gamma_{cde}^e + \Gamma_{e[cd]}^e) \\ &\quad + \left( X^{ac}Y^{bd} - Y^{ac}X^{bd} \right) \Gamma_{e(cd)}^e \\ &\quad + \frac{3}{2}\Gamma_{[cde]}^{[a}Y^{b]e}X^{cd} - \frac{3}{2}\Gamma_{[cde]}^{[a}X^{b]e}Y^{cd}, \end{aligned} \quad (3.35)$$

<sup>9</sup>In fact,  $\omega^* = 4 - N$  from (3.15) and (3.17).

which further reduces, with the ansatz (3.24), to

$$\begin{aligned}
 [X, Y]_G^{ab}(\nabla) - [X, Y]_G^{ab}(\partial) &= \frac{1}{2} \left( Y^{ab} X^{cd} - X^{ab} Y^{cd} \right) \left[ \left( \bar{\beta} - \frac{1}{N-2} \right) \Gamma_{cde}{}^e + \frac{N-4}{N-2} \Gamma_{e[cd]}{}^e \right] \\
 &\quad + \left( X^{ac} Y^{bd} - Y^{ac} X^{bd} \right) \Gamma_{e(cd)}{}^e.
 \end{aligned} \tag{3.36}$$

In order to meet the compatibility, each line should vanish separately. Hence, we require

$$\left( \bar{\beta} - \frac{1}{N-2} \right) \Gamma_{abc}{}^c + \frac{N-4}{N-2} \Gamma_{c[ab]}{}^c = 0, \quad \Gamma_{c(ab)}{}^c = 0, \tag{3.37}$$

which in fact coincide with (3.27) and (3.28). Thus, putting  $\bar{\beta} \equiv \frac{1}{2}$ , we re-derive (3.33) and, from (3.24), we arrive at the same conditions as before for the connection (3.34),

$$\Gamma_{[abc]}{}^d = 0, \quad \Gamma_{c(ab)}{}^c = 0. \tag{3.38}$$

### 3.3.3 Compatibility with the U-metric

Having fixed the total weight to be  $\frac{1}{2}(\frac{1}{2}p - \frac{1}{2}q + \omega)$  as (3.15), the compatibility of the semi-covariant derivative with the U-metric reads

$$\nabla_{ab} M_{cd} = \partial_{ab} M_{cd} - \frac{1}{2} \Gamma_{abe}{}^e M_{cd} + 2\Gamma_{ab(cd)} = 0. \tag{3.39}$$

Contracting  $c$  and  $d$  indices we get

$$\Gamma_{abe}{}^e = \frac{2}{N-4} \partial_{ab} \ln |M|. \tag{3.40}$$

Thus, the metric compatibility (3.39) is equivalent to

$$A_{abcd} := \Gamma_{ab(cd)} = -\frac{1}{2} \partial_{ab} M_{cd} + \frac{1}{2(N-4)} M_{cd} \partial_{ab} \ln |M|. \tag{3.41}$$

It is useful to note

$$\begin{aligned}
 A_{abe}{}^e &= \Gamma_{abe}{}^e = \frac{2}{N-4} \partial_{ab} \ln |M| = \frac{2}{N-4} M^{ef} \partial_{ab} M_{ef}, \\
 \partial_{ab} M_{cd} &= -2A_{abcd} + \frac{1}{2} A_{abe}{}^e M_{cd}, \\
 \partial_{ab} M^{cd} &= 2A_{ab}{}^{cd} - \frac{1}{2} A_{abe}{}^e M^{cd}.
 \end{aligned} \tag{3.42}$$

### 3.4 Determining the connection uniquely

Here, we derive the connection (2.18),

$$\begin{aligned}
 \Gamma_{abcd} &= A_{abcd} + \frac{1}{2} (A_{acbd} - A_{adbc} + A_{bdac} - A_{bcad}) \\
 &\quad + \frac{1}{N-2} (M_{ac} A^e{}_{(bd)e} - M_{ad} A^e{}_{(bc)e} + M_{bd} A^e{}_{(ac)e} - M_{bc} A^e{}_{(ad)e}),
 \end{aligned} \tag{3.43}$$

as the unique solution to the five constraints, (2.20), (2.21), (2.22), (2.23) and (2.24). The connection can be rewritten,

$$\Gamma_{abcd} = B_{[ab]cd} + \frac{1}{2} (B_{acbd} - B_{adbc} + B_{bdac} - B_{bcad}), \tag{3.44}$$

if we set

$$B_{abcd} := A_{abcd} + \frac{2}{N-2} M_{ab} A^e{}_{(cd)e}. \tag{3.45}$$

We start by recalling the five conditions for the connection,

$$\Gamma_{ab(cd)} = A_{abcd}, \quad (3.46)$$

$$\Gamma_{(ab)c}{}^d = 0, \quad (3.47)$$

$$\Gamma_{[abc]}{}^d = 0, \quad (3.48)$$

$$\Gamma_{c(ab)}{}^c = 0, \quad (3.49)$$

$$\mathcal{P}_{abcd}{}^{efgh} \Gamma_{efgh} = 0. \quad (3.50)$$

The first condition (3.46) is equivalent to the metric compatibility,  $\nabla_{ab} M_{cd} = 0$ , as discussed in section 3.3.3. The second condition (3.47) is natural, from the skew-symmetric property of the coordinates,  $x^{(ab)} = 0$  and hence  $\partial_{(ab)} = \nabla_{(ab)} = 0$ . The next two relations, (3.48) and (3.49), ensure the compatibilities with the generalized Lie derivative and also with the generalized bracket, as discussed in sections 3.3.1 and 3.3.2. The last condition (3.50) is a projection property which we deliberately impose in order to fix the connection uniquely. We may view the three constraints, (3.48), (3.49) and (3.50), as the ‘torsionless’ conditions. These are all — including the projection condition — analogous to the DFT-geometry of [66].

While the first condition, (3.46), fixes the symmetric part of the connection, the remaining ones should determine the anti-symmetric part,

$$\mathcal{X}_{abcd} = \mathcal{X}_{[ab][cd]} := \Gamma_{ab[cd]}, \quad (3.51)$$

satisfying

$$\Gamma_{abcd} = A_{abcd} + \mathcal{X}_{abcd}. \quad (3.52)$$

First of all, it follows from (3.46), (3.49),

$$\Gamma^c{}_{acb} + \Gamma^c{}_{bca} = 4A^c{}_{(ab)c}, \quad (3.53)$$

and also from (3.47), (3.49),

$$\Gamma^{cd}{}_{dc} = 0, \quad \Gamma^{cd}{}_{cd} = 0. \quad (3.54)$$

Further, from e.g.  $\Gamma_{[abc]d} - \Gamma_{[abd]c} + \Gamma_{[bcd]a} - \Gamma_{[acd]b} = 0$ , we get

$$\mathcal{X}_{abcd} - \mathcal{X}_{cdab} = 2A_{a[cd]b} - 2A_{b[cd]a}. \quad (3.55)$$

We then only need to determine

$$\mathcal{Y}_{abcd} := \frac{1}{2} (\mathcal{X}_{abcd} + \mathcal{X}_{cdab}) = \frac{1}{2} (\Gamma_{[ab][cd]} + \Gamma_{[cd][ab]}), \quad (3.56)$$

which satisfies, by construction, symmetric properties,

$$\mathcal{Y}_{abcd} = \mathcal{Y}_{[ab][cd]}, \quad \mathcal{Y}_{abcd} = \mathcal{Y}_{cdab}, \quad (3.57)$$

and contributes to the connection through

$$\Gamma_{abcd} = A_{abcd} + \frac{1}{2} (A_{acdb} - A_{adcb} - A_{bcda} + A_{bdca}) + \mathcal{Y}_{abcd}. \quad (3.58)$$



Now, all the constraints except the last one (3.50), boil down to

$$\mathcal{Y}_{[abc]d} = 0, \quad \mathcal{Y}^c_{acb} = A^c_{(ab)c}. \quad (3.59)$$

On the other hand, the last projection condition (3.50) fixes  $Y_{abcd}$  uniquely,

$$\mathcal{Y}_{abcd} = \frac{1}{N-2} (M_{ac}A^e_{(bd)e} - M_{bc}A^e_{(ad)e} + M_{bd}A^e_{(ac)e} - M_{ad}A^e_{(bc)e}). \quad (3.60)$$

It is straightforward to check for consistency that,  $\mathcal{Y}_{abcd}$  given in (3.60) indeed satisfies the relations (3.59) and also the (anti-)symmetric properties (3.57). Alternatively, one may well guess the expression (3.60) as a solution of (3.57) and (3.59), i.e. a solution that can be readily constructed in terms of the symmetric two-index objects,  $M_{ab}$  and  $A^e_{(ab)e}$ . The last condition (3.50) then ensures it to be the only solution.

Following the method in [30], the uniqueness can be also verified directly. First, it is straightforward to check that the connection given in (3.43) satisfies all the five conditions, (3.46)–(3.50). On the other hand, if the most general solution of them might contain an extra piece, say  $\Upsilon_{abcd}$ , the first four conditions, (3.46)–(3.49), imply

$$\Upsilon_{abcd} = \Upsilon_{[ab][cd]}, \quad \Upsilon_{[abc]d} = 0, \quad \Upsilon_{e(ab)}^e = 0, \quad (3.61)$$

such that in particular,  $\Upsilon_{[abc]}^a = \frac{2}{3}\Upsilon_{a[bc]}^a = 0$ . Consequently we get

$$\Upsilon_{eab}^e = 0. \quad (3.62)$$

The last condition (3.50) then reduces to

$$\Upsilon_{[ab][cd]} + \Upsilon_{[cd][ab]} = 0, \quad (3.63)$$

which further gives

$$\Upsilon_{abcd} = -\Upsilon_{cdab} = \Upsilon_{dacb} + \Upsilon_{acdb} = -\Upsilon_{bcad} - \Upsilon_{acbd} = \Upsilon_{abcd} - 2\Upsilon_{acbd}, \quad (3.64)$$

and hence, the verification of the uniqueness,

$$\Upsilon_{acbd} = 0. \quad (3.65)$$

To summarize, the five conditions, (3.46), (3.47), (3.48), (3.49) and (3.50), uniquely determines the connection (3.43).

### 3.5 Semi-covariant derivative and its complete covariantization

The infinitesimal diffeomorphic transformation of the U-metric,

$$\delta_X M_{ab} = \hat{\mathcal{L}}_X M_{ab} = \nabla_{ac} X_b^c + \nabla_{bc} X_a^c - \frac{1}{2} M_{ab} \nabla_{cd} X^{cd}, \quad (3.66)$$

induces upon the section condition,

$$\delta_X (\partial_{ab} M_{cd}) = \hat{\mathcal{L}}_X (\partial_{ab} M_{cd}) + M_{ed} \partial_{ab} \partial_{cf} X^{ef} + M_{ce} \partial_{ab} \partial_{df} X^{ef} - \frac{1}{2} M_{cd} \partial_{ab} \partial_{ef} X^{ef}, \quad (3.67)$$

and hence

$$\delta_X A_{abcd} = \hat{\mathcal{L}}_X A_{abcd} - \frac{1}{2} \left( \partial_{ab} \partial_{ce} X^{fe} \right) M_{fd} - \frac{1}{2} \left( \partial_{ab} \partial_{de} X^{fe} \right) M_{fc}. \quad (3.68)$$

It is then straightforward to derive the variation of the connection under diffeomorphism,

$$\delta_X \Gamma_{abc}{}^d = \hat{\mathcal{L}}_X \Gamma_{abc}{}^d - \partial_{ab} \partial_{ce} X^{de} + \mathcal{P}_{abc}{}^{dklm}{}_n \partial_{kl} \partial_{me} X^{ne}. \quad (3.69)$$

For consistency, this expression is compatible with all the properties of the connection, such as

$$\begin{aligned} \delta_X \Gamma_{(ab)c}{}^d &= \hat{\mathcal{L}}_X \Gamma_{(ab)c}{}^d \equiv 0, & \delta_X \Gamma_{[abc]}{}^d &= \hat{\mathcal{L}}_X \Gamma_{[abc]}{}^d \equiv 0, \\ \delta_X \Gamma_{c(ab)}{}^c &= \hat{\mathcal{L}}_X \Gamma_{cab}{}^c \equiv 0, & \delta_X \left( \mathcal{P}_{abcd}{}^{efgh} \Gamma_{efgh} \right) &= \hat{\mathcal{L}}_X \left( \mathcal{P}_{abcd}{}^{efgh} \Gamma_{efgh} \right) \equiv 0, \end{aligned} \quad (3.70)$$

which can be easily verified using e.g. the projection property ‘ $\mathcal{P}(1 - \mathcal{P}) = 0$ ’ (3.9) and an identity,

$$\partial_{c(a} \partial_{b)d} X^{cd} = 0. \quad (3.71)$$

Further, up to the section condition, we have

$$\delta_X \Gamma_{abe}{}^e = \hat{\mathcal{L}}_X \Gamma_{abe}{}^e - \partial_{ab} \partial_{ef} X^{ef}. \quad (3.72)$$

It is crucial to note that the last term in (3.69), which we put hereafter<sup>10</sup>

$$\Omega_{abc}{}^d := \mathcal{P}_{abc}{}^{dklm}{}_n \partial_{kl} \partial_{me} X^{ne}, \quad (3.73)$$

generates ‘anomalous’ terms in the variation of the semi-covariant derivative acting on a generic covariant tensor density,

$$\begin{aligned} \delta_X (\nabla_{ab} T^{c_1 c_2 \dots c_p}{}_{d_1 d_2 \dots d_q}) &= \hat{\mathcal{L}}_X (\nabla_{ab} T^{c_1 c_2 \dots c_p}{}_{d_1 d_2 \dots d_q}) \\ &\quad - \sum_{i=1}^p T^{c_1 \dots e \dots c_p}{}_{d_1 d_2 \dots d_q} \Omega_{abe}{}^{c_i} + \sum_{j=1}^q \Omega_{abd_j}{}^e T^{c_1 c_2 \dots c_p}{}_{d_1 \dots e \dots d_q}. \end{aligned} \quad (3.74)$$

The second line is the anomalous part. Hence, the semi-covariant derivative of a generic covariant tensor density is not necessarily covariant.<sup>11</sup> Nevertheless, from (3.9), (3.10), (3.11), (3.13) and (3.71),  $\Omega_{abcd}$  possesses some nice properties,

$$\Omega_{abcd} = \Omega_{[ab][cd]} = \Omega_{cdab}, \quad \Omega_{[abc]d} = 0, \quad \Omega_{acb}{}^c = 0, \quad \Omega_{abcd} = \mathcal{P}_{abcd}{}^{efgh} \Omega_{efgh}. \quad (3.75)$$

These ensure that, for consistency, the followings are exceptionally, fully covariant.

(i) The U-metric compatibility (3.46),

$$\nabla_{ab} M_{cd} = 0, \quad \delta_X (\nabla_{ab} M_{cd}) = \hat{\mathcal{L}}_X (\nabla_{ab} M_{cd}) = 0. \quad (3.76)$$

<sup>10</sup>In the case of  $N = 5$ ,  $\Omega_{abcd}$  coincides with ‘ $\frac{1}{4} H_{abcd}$ ’ in [30].

<sup>11</sup>This is also precisely analogous to DFT-geometry, cf. eq. (20) of [66], where the anomalous part in the diffeomorphic variation of the DFT semi-covariant derivative is dictated by a six-index projection operator.

(ii) Scalar density with an arbitrary extra weight,

$$\nabla_{ab}\phi = \partial_{ab}\phi + \frac{1}{2}\omega\Gamma_{abc}{}^c\phi, \quad \delta_X(\nabla_{ab}\phi) = \hat{\mathcal{L}}_X(\nabla_{ab}\phi). \quad (3.77)$$

(iii) Kronecker delta symbol,

$$\nabla_{ab}\delta_d^c = 0, \quad \delta_X(\nabla_{ab}\delta_d^c) = \hat{\mathcal{L}}_X(\nabla_{ab}\delta_d^c) = 0. \quad (3.78)$$

In particular, from (3.76) and (3.77), the integral measure,  $|M|^{\frac{1}{4-N}}$  having the extra weight,  $\omega = 1$ , is also covariantly constant,

$$\nabla_{ab}|M|^{\frac{1}{4-N}} = 0. \quad (3.79)$$

The key characteristic of the semi-covariant derivative is that, by (anti-)symmetrizing or contracting the  $\mathbf{SL}(N)$  vector indices in an appropriate manner, it can generate completely covariant derivatives acting on a generic covariant tensor density, (2.30),

$$\nabla_{[ab}T_{c_1c_2\cdots c_q]}, \quad (3.80)$$

$$\nabla_{ab}T^a, \quad (3.81)$$

$$\nabla^a{}_bT_{[ca]} + \nabla^a{}_cT_{[ba]}, \quad (3.82)$$

$$\nabla^a{}_bT_{(ca)} - \nabla^a{}_cT_{(ba)}, \quad (3.83)$$

$$\nabla_{ab}T^{[abc_1c_2\cdots c_q]} \quad (\text{divergence}), \quad (3.84)$$

$$\nabla_{ab}\nabla^{[ab}T^{c_1c_2\cdots c_q]} \quad (\text{Laplacian}). \quad (3.85)$$

Note that the nontrivial values of  $q$  in (3.80), (3.84) and (3.85) are restricted to  $q = 0, 1, 2, \dots, N-2$  only, since the anti-symmetrization of more than  $N$  number of  $\mathbf{SL}(N)$  vector indices is trivial.

Of course, from the U-metric compatibility,  $\nabla_{ab}M_{cd} = 0$ , the  $\mathbf{SL}(N)$  vector indices above can be freely raised or lowered without spoiling the full covariance. For example, the following is also fully covariant along with (3.80),

$$\nabla^{[ab}T^{c_1c_2\cdots c_q]}. \quad (3.86)$$

Especially, for the case of  $q = 0$ , the divergence (3.84) reads explicitly,

$$\nabla_{ab}T^{ab} = \partial_{ab}T^{ab} + \frac{1}{2}(\omega - 1)\Gamma_{abc}{}^cT^{ab}, \quad (3.87)$$

and hence,

$$\nabla_{ab}T^{ab} = \partial_{ab}T^{ab} \quad \text{for } \omega = 1. \quad (3.88)$$

This is a useful relation for the discussion of the ‘total derivative’ or ‘surface integral’ for the action.

Successive applications of the above procedure to a scalar and a vector — or directly from (3.91) — lead to the following second-order covariant derivatives,

$$\nabla_{[ab}\nabla_{cd]}\phi = 0, \quad \nabla_{[ab}\nabla_{cd}T_e] = 0, \quad \nabla_{[ab}\nabla_{c]d}T^d = 0, \quad (3.89)$$

which turn out to be all trivial due to (3.47), (3.48), (3.49) and the section condition. Similarly, for arbitrary a scalar and a vector, we have an identity,

$$\nabla_{[ab}\phi\nabla_{cd}T_{e]} = 0. \quad (3.90)$$

It is worth while to note, from (3.74),

$$\begin{aligned} & (\delta_X - \hat{\mathcal{L}}_X) (\nabla_{ab}\nabla_{cd}T^{e_1e_2\cdots e_p}_{f_1f_2\cdots f_q}) \\ &= -\sum_{i=1}^p (T^{e_1\cdots g\cdots e_p}_{f_1\cdots f_q}\nabla_{ab}\Omega_{cdg}{}^{e_i} + \nabla_{ab}T^{e_1\cdots g\cdots e_p}_{f_1\cdots f_q}\Omega_{cdg}{}^{e_i} + \nabla_{cd}T^{e_1\cdots g\cdots e_p}_{f_1\cdots f_q}\Omega_{abg}{}^{e_i}) \\ &+ \sum_{j=1}^q (\nabla_{ab}\Omega_{cdf_j}{}^g T^{e_1\cdots e_p}_{f_1\cdots g\cdots f_q} + \Omega_{abf_j}{}^g \nabla_{cd}T^{e_1\cdots e_p}_{f_1\cdots g\cdots f_q} + \Omega_{cdf_j}{}^g \nabla_{ab}T^{e_1\cdots e_p}_{f_1\cdots g\cdots f_q}) \\ &+ \Omega_{abc}{}^g \nabla_{gd}T^{e_1\cdots e_p}_{f_1\cdots f_q} + \Omega_{abd}{}^g \nabla_{cg}T^{e_1\cdots e_p}_{f_1\cdots f_q}. \end{aligned} \quad (3.91)$$

Further, from (2.24) and (3.75), we have

$$\Omega^{abcd}\Gamma_{abcd} = 0, \quad (3.92)$$

which also implies with (2.21) and (3.75),

$$\Omega^{abcd}\Gamma_{acbd} = 0. \quad (3.93)$$

### 3.6 Semi-covariant Riemann curvature and its complete covariantization

The commutator of the semi-covariant derivative leads to an expression,

$$\begin{aligned} & [\nabla_{ab}, \nabla_{cd}] T^{e_1\cdots e_p}_{f_1\cdots f_q} \\ &= -\sum_{i=1}^p T^{e_1\cdots g\cdots e_p}_{f_1\cdots f_q} R_{abcdg}{}^{e_i} + \sum_{j=1}^q R_{abcdf_j}{}^g T^{e_1\cdots e_p}_{f_1\cdots g\cdots f_q} \\ &+ \left( 2\Gamma_{ab[c}{}^g \delta_{d]}{}^h - 2\Gamma_{cd[a}{}^g \delta_{b]}{}^h - \frac{1}{2}\Gamma_{abk}{}^k \delta_c{}^g \delta_d{}^h + \frac{1}{2}\Gamma_{cdk}{}^k \delta_a{}^g \delta_b{}^h \right) \nabla_{gh} T^{e_1\cdots e_p}_{f_1\cdots f_q}, \end{aligned} \quad (3.94)$$

where  $R_{abcde}{}^f$  denotes the standard “field strength” of the connection,

$$\begin{aligned} R_{abcde}{}^f &:= \partial_{ab}\Gamma_{cde}{}^f - \partial_{cd}\Gamma_{abe}{}^f + \Gamma_{abe}{}^g \Gamma_{cdg}{}^f - \Gamma_{cde}{}^g \Gamma_{abg}{}^f, \\ &= \nabla_{ab}\Gamma_{cde}{}^f + \frac{1}{2}\Gamma_{abg}{}^g \Gamma_{cde}{}^f + \Gamma_{cde}{}^g \Gamma_{abg}{}^f - \Gamma_{abc}{}^g \Gamma_{gde}{}^f - \Gamma_{abd}{}^g \Gamma_{cge}{}^f - [(a, b) \leftrightarrow (c, d)], \end{aligned} \quad (3.95)$$

which we call henceforth *the fake curvature*. The fake curvature satisfies identities that are rather trivial,

$$R_{abcde}{}^f + R_{cdabe}{}^f = 0, \quad R_{[abcd]e}{}^f = 0. \quad (3.96)$$

On the other hand, from  $[\nabla_{ab}, \nabla_{cd}] M_{ef} = 0$  for (3.95), nontrivial identities are

$$R_{abcdef} + R_{abdcfe} = 0, \quad (3.97)$$

and hence, we get<sup>12</sup>

$$R_{abcdef} = R_{[ab][cd][ef]} = -R_{[cd][ab][ef]}. \quad (3.98)$$

In particular, the fake curvature is traceless,

$$R_{abcde}{}^e = 0. \quad (3.99)$$

We define the *semi-covariant Riemann curvature*,

$$\begin{aligned} S_{abcd} := & 3\partial_{[ab}\Gamma_{e][cd]}{}^e + 3\partial_{[cd}\Gamma_{e][ab]}{}^e + \frac{1}{4}\Gamma_{abe}{}^e\Gamma_{cdf}{}^f + \frac{1}{2}\Gamma_{abe}{}^f\Gamma_{cdf}{}^e \\ & + \Gamma_{ab[c}{}^e\Gamma_{d]ef}{}^f + \Gamma_{cd[a}{}^e\Gamma_{b]ef}{}^f + \Gamma_{ea[c}{}^f\Gamma_{d]fb}{}^e - \Gamma_{eb[c}{}^f\Gamma_{d]fa}{}^e. \end{aligned} \quad (3.100)$$

The semi-covariant Riemann curvature can be rewritten, using the semi-covariant derivative,

$$\begin{aligned} S_{abcd} = & 3\nabla_{[ab}\Gamma_{e][cd]}{}^e + 3\nabla_{[cd}\Gamma_{e][ab]}{}^e - \frac{1}{4}\Gamma_{abe}{}^e\Gamma_{cdf}{}^f - \frac{1}{2}\Gamma_{abe}{}^f\Gamma_{cdf}{}^e \\ & - \Gamma_{ab[c}{}^e\Gamma_{d]ef}{}^f - \Gamma_{cd[a}{}^e\Gamma_{b]ef}{}^f - \Gamma_{ea[c}{}^f\Gamma_{d]fb}{}^e + \Gamma_{eb[c}{}^f\Gamma_{d]fa}{}^e, \end{aligned} \quad (3.101)$$

or in terms of the fake curvature,

$$\begin{aligned} S_{abcd} = & R_{abe[cd]}{}^e + R_{cde[ab]}{}^e - \frac{1}{2}\Gamma_{abe}{}^f\Gamma_{cdf}{}^e + \frac{1}{4}\Gamma_{abe}{}^e\Gamma_{cdf}{}^f \\ & + \frac{1}{2}\Gamma_{ead}{}^f\Gamma_{fcb}{}^e + \frac{1}{2}\Gamma_{eda}{}^f\Gamma_{fbc}{}^e - \frac{1}{2}\Gamma_{ebd}{}^f\Gamma_{fca}{}^e - \frac{1}{2}\Gamma_{edb}{}^f\Gamma_{fac}{}^e \\ & + \frac{1}{4}\Gamma_{eaf}{}^f\Gamma_{cdb}{}^e + \frac{1}{4}\Gamma_{ecf}{}^f\Gamma_{abd}{}^e - \frac{1}{4}\Gamma_{ebf}{}^f\Gamma_{cda}{}^e - \frac{1}{4}\Gamma_{edf}{}^f\Gamma_{abc}{}^e. \end{aligned} \quad (3.102)$$

By construction, it satisfies symmetric properties,

$$S_{abcd} = S_{[ab][cd]}, \quad S_{abcd} = S_{cdab}, \quad (3.103)$$

and thanks to the section condition, it meets the Bianchi identity,

$$S_{[abc]d} = 0. \quad (3.104)$$

Under arbitrary variation of the connection,  $\delta\Gamma_{abc}{}^d$ , which is, from (2.21), (2.22), (2.23), subject to

$$\delta\Gamma_{(ab)c}{}^d = 0, \quad \delta\Gamma_{[abc]}{}^d = 0, \quad \delta\Gamma_{c(ab)}{}^c = 0, \quad (3.105)$$

the fake curvature transforms in a somewhat complicated manner,

$$\delta R_{abcde}{}^f = \nabla_{ab}\delta\Gamma_{cde}{}^f + \frac{1}{2}\Gamma_{abg}{}^g\delta\Gamma_{cde}{}^f - \Gamma_{abc}{}^g\delta\Gamma_{gde}{}^f - \Gamma_{abd}{}^g\delta\Gamma_{cge}{}^f - [(a, b) \leftrightarrow (c, d)]. \quad (3.106)$$

On the other hand, the semi-covariant Riemann curvature transforms as total derivative,

$$\delta S_{abcd} = 3\nabla_{[ab}\delta\Gamma_{e][cd]}{}^e + 3\nabla_{[cd}\delta\Gamma_{e][ab]}{}^e. \quad (3.107)$$

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<sup>12</sup>Eq. (3.98) implies that there exists essentially *only* one fake ‘scalar’ curvature one can construct by contracting the indices of  $R_{abcdef}$ , which is  $R_{abc}{}^{abc}$  [30].

In fact, this is the crucial ‘defining’ property of the semi-covariant Riemann curvature which we preredquired to derive the expression (3.100).

Especially, under diffeomorphism (3.69), while the connection changes,

$$\delta_X \Gamma_{abc}{}^d = \hat{\mathcal{L}}_X \Gamma_{abc}{}^d - \partial_{ab} \partial_{ce} X^{de} + \Omega_{abc}{}^d, \quad \Omega_{abcd} = \mathcal{P}_{abcd}{}^{klm}{}_n \partial_{kl} \partial_{me} X^{ne}, \quad (3.108)$$

the fake curvature varies,

$$\begin{aligned} \delta_X R_{abcdef} - \hat{\mathcal{L}}_X R_{abcdef} = & \nabla_{ab} \Omega_{cdef} + \frac{1}{2} \Gamma_{abg}{}^g \Omega_{cdef} - \Gamma_{abc}{}^g \Omega_{gdef} - \Gamma_{abd}{}^g \Omega_{gcef} \\ & + \partial_{ab} \partial_{ch} X^{gh} \Gamma_{gd[ef]} + \partial_{ab} \partial_{dh} X^{gh} \Gamma_{cg[ef]} - \frac{1}{2} \partial_{ab} \partial_{gh} X^{gh} \Gamma_{cd[ef]} \\ & - [(a, b) \leftrightarrow (c, d)], \end{aligned} \quad (3.109)$$

and the semi-covariant Riemann curvature transforms *neatly*,

$$\delta_X S_{abcd} = \hat{\mathcal{L}}_X S_{abcd} + 2 \nabla_{[a} \Omega_{b][cd]}{}^e + 2 \nabla_{e[c} \Omega_{d][ab]}{}^e. \quad (3.110)$$

Like the semi-covariant derivative (3.74), the anomalous terms are dictated by the projection operator.<sup>13</sup> Therefore, as the name indicates, the fake curvature,  $R_{abcdef}$ , is not covariant. Yet, with the nice properties of  $\Omega_{abcd}$  (3.75), the semi-covariant Riemann curvature can be completely covariantized, such as Ricci and scalar curvatures:<sup>14</sup>

$$\begin{aligned} S_{ab} &:= S_{acb}{}^c = S_{ba}, & \delta_X S_{ab} &= \hat{\mathcal{L}}_X S_{ab}, \\ S &:= M^{ab} S_{ab} = S_{ab}{}^{ab}, & \delta_X S &= \hat{\mathcal{L}}_X S = \frac{1}{2} X^{ab} \partial_{ab} S. \end{aligned} \quad (3.111)$$

For later use, it is worth while to have an explicit expression of the completely covariant scalar curvature,

$$S = -2 \partial_{ab} (2 A^{cab}{}_c + A^{abc}{}_c) + A_{abcd} A^{abcd} - 4 A_{abcd} A^{acbd} - \frac{1}{2} A_{abc}{}^c A^{abd}{}_d - 4 A_{cab}{}^c A^{abd}{}_d + 4 A_{cab}{}^c A^{dba}{}_d, \quad (3.112)$$

where, as defined before (3.41),

$$A_{abcd} = -\frac{1}{2} \partial_{ab} M_{cd} + \frac{1}{2(N-4)} M_{cd} \partial_{ab} \ln |M|. \quad (3.113)$$

### 3.7 Action and the Einstein equation of motion

From (3.107), it is straightforward to derive the variation of the fully covariant scalar curvature,

$$\delta S = 2 \delta M^{ab} S_{ab} + 6 \nabla_{[ab} \left( \delta \Gamma_{e]cd}{}^e M^{ac} M^{bd} \right). \quad (3.114)$$

Hence, disregarding surface integral, arbitrary variation of the U-metric induces the following transformation of the U-gravity action (2.32),

$$\delta \left( \int_{\Sigma} M^{\frac{1}{4-N}} S \right) = \int_{\Sigma} M^{\frac{1}{4-N}} \delta M^{ab} \left( 2 S_{ab} + \frac{1}{N-4} M_{ab} S \right), \quad (3.115)$$

which leads to the *Einstein equation of motion* (2.33). Further, from the invariance of the action under diffeomorphism (3.66), a *conservation relation* (2.34) follows.<sup>15</sup>

<sup>13</sup>Again, this is precisely analogous to the DFT-geometry, cf. eq. (27) in ref. [66].

<sup>14</sup>Note that  $S_{ab}$  and  $S$  are related to ‘ $\mathcal{R}_{ab}$ ’ and ‘ $\mathcal{R}$ ’ in [30] by factor two:  $S_{ab} = 2\mathcal{R}_{ab}$ ,  $S = 2\mathcal{R}$ .

<sup>15</sup>The *conservation relation* (2.34) may be also directly verified using the Jacobiator of the semi-covariant derivative, cf. eq. (4.3) in ref. [30].

### 3.8 Reductions

Here we discuss the reduction of  $\mathbf{SL}(N)$  U-gravity upon each section,  $\Sigma_{N-1}$  and  $\Sigma_3$  separately. The resulting gravitational actions contain  $(N-2)$ -form or two-form potentials as well as scalars.<sup>16</sup>

1. *Reduction upon  $\Sigma_{N-1}$ .* In order to perform the Riemannian reduction to the  $(N-1)$ -dimensional section,  $\Sigma_{N-1}$  (2.35), we parametrize the U-metric by [22, 30]

$$M_{ab} = \begin{pmatrix} \frac{g_{\alpha\beta}}{\sqrt{|g|}} & v_\alpha \\ v_\beta & \sqrt{|g|}(-e^\phi + v^2) \end{pmatrix}, \quad M^{ab} = \begin{pmatrix} \sqrt{|g|}(g^{\alpha\beta} - e^{-\phi}v^\alpha v^\beta) & e^{-\phi}v^\alpha \\ e^{-\phi}v^\beta & -\frac{e^{-\phi}}{\sqrt{|g|}} \end{pmatrix}. \quad (3.116)$$

Here  $\phi$ ,  $v^\alpha$  and  $g_{\alpha\beta}$  denote a scalar, a vector and a Riemannian metric on  $\Sigma_{N-1}$ , such that  $v_\alpha = g_{\alpha\beta}v^\beta$ ,  $v^2 = g^{\alpha\beta}v_\alpha v_\beta$  and  $g = \det(g_{\alpha\beta})$ . The vector can be freely dualized to an  $(N-2)$ -form potential which may couple to an  $(N-3)$ -brane.

With the Riemannian ansatz (3.116), the U-gravity scalar curvature (3.112) reduces to (2.40) which agrees with [30] when  $N = 5$ . Consistently, the generalized Lie derivative (2.8) decomposes into the  $(N-1)$ -dimensional ordinary Lie derivative and the gauge symmetry of the  $(N-2)$ -form potential. We refer the readers to eq. (5.6) of [30] for the explicit demonstration in the case of  $\mathbf{SL}(5)$ .

It is crucial to note that a nontrivial assumption has been implicitly made to write the ansatz (3.116), namely that the upper left  $(N-1) \times (N-1)$  block of the U-metric is *non-degenerate* and hence we are allowed to write “ $g_{\alpha\beta}/\sqrt{|g|}$ ” there. However, the rank of the  $(N-1) \times (N-1)$  block can be  $N-2$  (but not less than that for the U-metric to be non-degenerate). The degenerate case then corresponds to a non-Riemannian background where the Riemannian metric ceases to exist. Nevertheless,  $\mathbf{SL}(N)$  U-gravity has no problem with that. One example of such a non-Riemannian background is given by a U-metric of which the only nontrivial components are  $M_{1N} = M_{N1}$  and  $M_{\hat{\alpha}\hat{\beta}}$  with  $2 \leq \hat{\alpha}, \hat{\beta} \leq N-1$ .

2. *Reduction upon  $\Sigma_3$ .* For the Riemannian reduction of U-gravity to the three-dimensional section,  $\Sigma_3$  (2.36), we put [80],

$$M_{ab} = \begin{pmatrix} \sqrt{|\tilde{g}|}(\tilde{g}_{\mu\nu} + e^{\tilde{\phi}}\tilde{v}^k{}_\mu\tilde{v}_{k\nu}) & e^{\tilde{\phi}}\tilde{v}_{j\mu} \\ e^{\tilde{\phi}}\tilde{v}_{i\nu} & \frac{e^{\tilde{\phi}}}{\sqrt{|\tilde{g}|}}\tilde{\mathcal{M}}_{ij} \end{pmatrix}, \quad M^{ab} = \begin{pmatrix} \frac{\tilde{g}^{\mu\nu}}{\sqrt{|\tilde{g}|}} & -\tilde{v}^{j\mu} \\ -\tilde{v}^{i\nu} & \sqrt{|\tilde{g}|}(e^{-\tilde{\phi}}\tilde{\mathcal{M}}^{ij} + \tilde{v}^{i\lambda}\tilde{v}^j{}_\lambda) \end{pmatrix}. \quad (3.117)$$

Here, to be consistent with the ‘lower’ index of the dual coordinates,  $\tilde{x}_\mu$ , the Riemannian metric is  $\tilde{g}^{\mu\nu}$  having ‘upper’ indices, with the determinant,  $\tilde{g} \equiv \det(\tilde{g}^{\mu\nu})$ ;  $\tilde{\mathcal{M}}_{ij}$  is a symmetric  $(N-3) \times (N-3)$  unit determinant matrix; and  $\tilde{v}_{i\mu}$  are  $(N-3)$  copies of vectors while  $\tilde{v}^{i\mu} = \tilde{\mathcal{M}}^{ij}\tilde{g}^{\mu\nu}\tilde{v}_{j\nu}$ . The vectors can be dualized to two-form potentials.

<sup>16</sup>Hence, *a priori*, the  $\mathbf{SL}(N)$  duality differs from the Ehlers group of the Einstein-Hilbert pure gravity action, cf. [35].

With the Riemannian ansatz (3.117), the U-gravity scalar curvature (3.112) reduces to (2.42) which features  $\mathbf{SL}(N-3)$  S-duality and agrees with [80] when  $N = 5$ . Consistently, the generalized Lie derivative (2.8) decomposes into the three-dimensional ordinary Lie derivative and the gauge symmetry of the two-form potentials. We refer the readers to eq. (3.8) of [80] for the explicit demonstration in the case of  $\mathbf{SL}(5)$ .

Writing (3.117), it has been assumed that the upper left  $3 \times 3$  block of  $M^{ab}$  is non-degenerate. But, in general, its rank can be less than 3. In fact, when  $N \geq 6$  the whole block can vanish: for example the only nontrivial components of the inverse of the U-metric can be,  $M^{\mu\hat{i}} = M^{\hat{i}\mu}$  and  $M^{\tilde{i}\tilde{j}}$  where  $\hat{i} = 4, 5, 6$  and  $7 \leq \tilde{i}, \tilde{j} \leq N$ . When  $N = 5$ , the rank of the  $3 \times 3$  block is either 3 (non-degenerate) or at least 2 (degenerate).

## 4 Outlook

On the extended-yet-gauged spacetime, the usual differential one-form of the coordinate,  $dx^{ab}$ , is not invariant under the coordinate gauge symmetry (2.3), and thus needs to be *gauged*, cf. [30]

$$Dx^{ab} := dx^{ab} - A^{ab}, \quad A^{ab}\partial_{ab} \equiv 0. \quad (4.1)$$

Here a connection has been introduced which assumes the same ‘value’ as the coordinate gauge symmetry generator, or the shift (2.4). Essentially it gauges away the orthogonal directions to a chosen section. The gauged one-form can be then used to construct an  $\mathbf{SL}(N)$  duality manifest world-volume action for an  $(N-3)$ -brane propagating in the extended-yet-gauged spacetime, as done for a string in [30] (cf. [82–84]).

The notion of the cosmological ‘constant’ depends on the kind of differential geometry in use [66]. In  $\mathbf{SL}(N)$  U-gravity, the natural cosmological constant term reads

$$\int_{\Sigma} M^{\frac{1}{4-N}} \Lambda. \quad (4.2)$$

Yet, from the Riemannian point of view, i.e. (2.39) or (2.41), this term corresponds to an exponential potential of the scalar. This might provide a new spin on the cosmological constant problem, cf. [85–87].

Recent studies indicate that, in order to identify the DFT/EFT origins of all the known lower dimensional gauged supergravities, it is necessary to ‘relax’ the section condition [31, 41, 78, 88–94]. The geometric insight of the extended-yet-gauged spacetime is then somewhat unclear. Perhaps, the strict invariance under the coordinate gauge symmetry (2.4) may not be the only way to realize the extended-yet-gauged spacetime. The final geometric understanding is incomplete. In this line, it is worth while to note an interesting recent development [59] where the flat  $\mathbf{O}(D, D)$  metric in DFT is promoted to a generic curved one and the section condition is accordingly modified.

Understanding of the global and topological aspects of  $\mathbf{SL}(N)$  U-gravity is desirable along with further geometric insights into the non-Riemannian backgrounds, cf. [54, 56–60, 95, 96].



Taking  $N = 11$ ,  $\mathbf{SL}(11)$  U-gravity may provide an  $\mathbf{SL}(11)$  U-duality manifest reformulation of the ten-dimensional massive type IIA supergravity [97] with the identification of the ten-form flux as the cosmological constant [98]. This will be in analogous to the  $\mathbf{O}(10, 10)$  T-duality manifest unification of IIA and IIB supergravities [71] (cf. [72, 73]).

In view of the Dynkin diagram (table 1), putting  $\mathbf{SL}(11)$  U-gravity and  $\mathbf{O}(10, 10)$  DFT together, one may anticipate the whole  $E_{11}$  structure to emerge. A tantalizing clue comes from the RR nine-form potential, which is dual to the vector in the  $\Sigma_{10}$  parametrization of the U-metric (2.39). In the  $\mathcal{N} = 2$   $D = 10$  SDFT of [71], the local Lorentz group is doubled to be  $\mathbf{Spin}(1, 9)_L \times \mathbf{Spin}(1, 9)_R$  and the whole RR-sector is represented by a single  $\mathbf{Spin}(1, 9)_L \times \mathbf{Spin}(1, 9)_R$  bi-spinorial object which is *a priori*  $\mathbf{O}(10, 10)$  singlet. After diagonal gauge fixing of the doubled local Lorentz group, the single bi-spinorial object may decompose into various RR  $p$ -form potentials which are no longer  $\mathbf{O}(10, 10)$  singlet but form an  $\mathbf{O}(10, 10)$  spinor, to agree with [72, 73]. On the other hand, in  $\mathbf{SL}(11)$  U-gravity, the  $\mathbf{SL}(11)$  group does not mix the RR nine-form potential with other RR  $p$ -form potentials, since only the nine-form potential enters the parametrization of the U-metric (2.39). This might shed light on the  $E_{11}$  duality manifest reformulation of the maximal supergravity. But here we can only speculate.

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